ISRAEL JOURNAL OF MATHEMATICS **94** (1996), 1–5

A FOOTNOTE ON RESIDUALLY FINITE GROUPS

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ABSTRACT

We give an easy proof that a finitely generated group which is residually (finite and soluble of bounded rank) is nilpotent by quasi-linear. This can be used to shorten the proofs of some recent theorems about residually finite groups.

In 1988 Lubotzky [Lu] enunciated a pretty necessary and sufficient condition for a finitely generated group to have a faithful linear representation in characteristic zero. What this condition amounts to is that the group should have a normal subgroup G of finite index such that, for some prime p, some faithful pro-p completion of G has finite rank as a pro-p group. The proof of sufficiency rests on Lazard's theory of p-adic analytic groups [La], together with the theorem [LM1] that pro-p groups of finite rank are virtually powerful (and hence, according to Lazard, p-adic analytic).

The sufficiency of Lubotzky's criterion has been a cornerstone in the proofs of several recent theorems about residually finite groups, for example the characterisation of finitely generated groups with polynomial subgroup growth [LMS]. The philosophy of such proofs is outlined in Mann's survey [M], and a self-contained account of the background theory is given in [DDMS]. Indeed, it was the existence of such applications which motivated the authors of [DDMS] to produce the book.

The purpose of this note is to present an alternative to Lubotzky's criterion. This "weak linearity lemma", while giving much less precise information, nonetheless can serve just as well in applications of the sort mentioned above.

Received January 16, 1995

Thus the theory of analytic pro-p groups turns out, in this context, to be redundant. (As an author of [DDMS], I hasten to point out that the theory has great interest in its own right, as well as other applications in group theory — see [DDMS] Chapter 6 — and number theory.)

The use of this kind of argument as a tool in the characterisation of residually finite groups of finite rank originates with N. S. Chernikov [Ch]. My result is essentially a distillation of ideas from that paper and from Chapter 13 of Wehrfritz's book [W]; nonetheless, my hope is that its simple statement and short, easy proof will make it a useful alternative to the theory of analytic pro-p groups in certain group-theoretic investigations.

THEOREM: *Let G be a finitely generated group, and suppose that G is residually (finite and soluble of bounded rank). Then G has a nilpotent normal subgroup Q such that G/Q is a subdirect product of finitely many linear groups over fields.*

If, moreover, every finite quotient of G is soluble, then G is virtually nilpotent-by-abelian.

Here, by the **rank** of a finite group H is meant the least integer r such that every subgroup of H can be generated by r elements. We use the following notation: $C_G(X)$ denotes the centraliser in G of X; $[A, B] = [A,_1 B]$ denotes the group generated by all commutators $[a, b] = a^{-1}a^b \quad (a \in A, b \in B)$; $[A,_k B] = [[A,_{k-1} B], B]$ for $k > 1$; $\gamma_k(G) = [G,_{k-1} G]$ and $G' = \gamma_2(G)$.

Proof: Let G/K be a finite soluble quotient of G. Then G/K has a normal subgroup N/K which is nilpotent of class at most 2 and satisfies $C_G(N/K) \leq N$ (see [S] Chapter 2, Proposition 3). Put $E = N/N'K$.

CLAIM: If $H \lhd G$ and $[E,_k H] = 1$ then $\gamma_{6k} H \leq K$.

Proof: Write $\bar{N} = N/K$. Then

$$\begin{aligned}
[\bar{N},_k H] \leq \bar{N}' &\implies [\bar{N}',_{2k} H] \leq \gamma_3 \bar{N} = 1 \\
&\implies [\bar{N},_{3k} H] = 1 \\
&\implies \gamma_{3k} H \leq C_G(\bar{N}) \leq N \\
&\implies \gamma_{6k} H \leq [N,_{3k} H] \leq K
\end{aligned}$$

(cf. [S], Chapter 1, Prop. 14 and Prop. 10).

Now for some finite r, G has a family $\mathcal{S}$ of normal subgroups, intersecting in the identity, such that G/K is finite and soluble of rank at most r for each

$K \in \mathcal{S}$. For $K \in \mathcal{S}$ let E_K be the section of G/K indicated above, and write Z for the Cartesian product of all the abelian groups E_K. Then Z is an r-generator module for the ring $R = \mathbb{Z}^{\mathcal{S}}$, on which G acts by R-module automorphisms. Since G is finitely generated, there exist a finitely generated subring S of R and an r-generator S-submodule M of Z such that $MG = M$ and $MR = Z$.

Since S is a commutative Noetherian ring, M contains a finite chain of fully invariant S-submodules

$$0 = M_0 < M_1 < \cdots < M_k = M$$

such that, for each j, M_j/M_{j-1} is a finitely generated torsion-free S/P_j-module, where $P_j = \mathrm{ann}_S(M_j/M_{j-1})$ is a prime ideal of S ([W], Lemma 13.2). Put $Q_j = \mathrm{C}_G(M_j/M_{j-1})$, and suppose that M_j/M_{j-1} can be generated by r_j elements as an S/P_j-module. Then the action of G embeds G/Q_j in

$$\mathrm{Aut}_{S/P_j}(M_j/M_{j-1}) \leq \mathrm{GL}_{r_j}(F_j)$$

where F_j is the field of fractions of S/P_j.

Put $Q = Q_1 \cap \cdots \cap Q_k$. Then

$$Z(Q-1)^k = M(Q-1)^k R = 0.$$

It follows that $[E_K, {}_kQ] = 1$ for every $K \in \mathcal{S}$. By the initial *Claim*, this implies that

$$\gamma_{6k} Q \leq \bigcap \mathcal{S} = 1.$$

This proves the first part of the theorem.

Suppose now that every finite quotient of G is soluble. Let $1 \leq j \leq k$, put $S_j = S/P_j$ and $V_j = M_j/M_{j-1}$. If L is a maximal ideal of S_j, then G induces on V_j/V_jL a finite linear group of degree at most r_j. It follows by Mal'cev's theorem ([W], Theorem 3.6 or [S], Chapter 2, Theorem 3) that G has a normal subgroup H, of finite index bounded by a function of r_j, such that H' acts unipotently on V_j/V_jL. Since G is finitely generated, we can choose $H = H_j$, say, independently of L, and then have

$$V_j(H_j' - 1)^{r_j} \subseteq \bigcap_L V_j L = 0,$$

(for the last equality see the *Remark* below).

Now let $T = (\bigcap_{j=1}^{k} H_j)'$ and put $s = r_1 + \cdots + r_k$. Then $M(T-1)^s = 0$. As above, this implies that $\gamma_{6s}T = 1$, and the second part of the theorem follows.

To illustrate how this is applied, let me sketch the (now very short) proof of Theorem A of [MS]: *a finitely generated residually finite group of finite upper rank is virtually soluble of finite rank.* Let G be such a group. By Theorem O of [MS] (a result taken from [LM2]), G has a normal subgroup G_1 of finite index such that every finite quotient of G_1 is soluble. By the Theorem above, G_1 is virtually nilpotent-by-abelian; and the result follows by [MS] Lemma 2.2.

The result just proved forms one of the steps in the argument of [LMS], establishing the characterisation of groups with polynomial subgroup growth. It is the only step which involves the theory of pro-p groups (although, as a matter of history, the characterisation of pro-p groups with polynomial subgroup growth [LM3] was of great heuristic importance, as one of the first steps on the road to the full result of [LMS]).

Remark: It seems difficult to find a reference for the following elementary fact of commutative algebra: *If S is a finitely generated integral domain and V is a finitely generated torsion-free S-module, then $\bigcap_L VL = 0$, where L ranges over all maximal ideals of S.* Here is the proof. Let F be a maximal free submodule of V. Then $Vx \subseteq F$ for some non-zero $x \in S$, since V/F is finitely generated and torsion. Put $D = \bigcap_L VL$. We have

$$Dx \subseteq \bigcap_L FL = 0,$$

since S has zero Jacobson radical (see e.g. [N], Chapter 6, Theorem 1); therefore $D = 0$ since V is torsion-free.

References

[Ch] N. S. Chernikov, *Theorem on groups of finite special rank*, Ukrainskii Matematicheskii Zhurnal **42** (1990), 962–970.

[DDMS] J. D. Dixon, M.P.F. du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, LMS Lecture Notes 157, Cambridge, 1991.

[La] M. Lazard, *Groupes analytiques p-adiques*, Publications Mathématiques de l'Institut Hautes Études Scientifiques **26** (1965), 389–603.

[Lu] A. Lubotzky, *A group-theoretic characterisation of linear groups*, Journal of Algebra **113** (1988), 207–214.

[LM1] A. Lubotzky and A. Mann, *Powerful p-groups I, II,* Journal of Algebra **105** (1987), 484–515.

[LM2] A. Lubotzky and A. Mann, *Residually finite groups of finite rank,* Mathematical Proceedings of the Cambridge Philosophical Society **106** (1989), 385–388.

[LM3] A. Lubotzky and A. Mann, *On groups of polynomial subgroup growth,* Inventiones mathematicae **104** (1991), 521–533.

[LMS] A. Lubotzky, A. Mann and D. Segal, *Finitely generated groups of polynomial subgroup growth,* Israel Journal of Mathematics **82** (1993), 363–371.

[M] A. Mann, *Some applications of powerful p-groups,* in *Groups—St Andrews 1989,* Vol. 2, LMS Lecture Notes 160, Cambridge, 1991, pp. 370–385.

[MS] A. Mann and D. Segal, *Uniform finiteness conditions in residually finite groups,* Proceedings of the London Mathematical Society **61** (1990), 529–545.

[N] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities,* CUP, Cambridge, 1968.

[S] D. Segal, *Polycyclic Groups,* CUP, Cambridge, 1983.

[W] B. A. F. Wehrfritz, *Infinite Linear Groups,* Springer-Verlag, Berlin, 1973.